CHAPTER 7: THE DE FINETTI THEOREM

§ 7.1 Extendibility of quantum states

Consider a bipartite state ρ_{AB}. We call ρ_{AB} k-extendible if there exists a k-extension, a state $\rho_{AB_1\ldots B_k}$ with $B_i \cong B$ and $\rho_{AB_i} = \mathrm{tr}_{B_1\ldots B_{i-1}B_{i+1}\ldots B_k}\, \rho_{AB_1\ldots B_k} = \rho_{AB}$ for all $i = 1, \ldots, k$.

$\boxed{\text{Lem}}$ Separable states are ∞-extendible.

Proof: Let $\sigma_{AB} = \sum_i p_i\, \sigma_A^{(i)} \otimes \sigma_B^{(i)}$ be separable, then

$$\sigma_{AB_1\ldots B_k} = \sum_i p_i\, \sigma_A^{(i)} \otimes \sigma_{B_1}^{(i)} \otimes \ldots \otimes \sigma_{B_k}^{(i)} \quad \text{defines a}$$

k-extension for arbitrary $k \in \mathbb{N}$. □

Conversely, one can show that for every entangled state ρ_{AB} there exists a k_0 such that ρ_{AB} has no k-extension for $k \geq k_0$.

Ex.: Pure entangled states are not even 2-extendible.

This is usually called monogamy of entanglement:

A quantum system cannot be entangled with a large number of other systems.

De Finetti theorems provide a quantitative version of monogamy.

§ 7.2 A De Finetti theorem for pure symmetric states

We will focus on pure states in the symmetric subspace:

$$|\psi\rangle \in \mathrm{Sym}^n(\mathbb{C}^d) = \{ |\psi\rangle \in (\mathbb{C}^d)^{\otimes n} : \varphi(\pi)|\psi\rangle = |\psi\rangle \ \forall \pi \in S_n \}$$

Note that $\dim \mathrm{Sym}^n(\mathbb{C}^d) = \binom{n+d-1}{n}$ by Weyl's dim. formula

$\boxed{\text{Lem}}$ Let $|\varphi\rangle \in \mathbb{C}^d$ be arbitrary. Then

$$\Pi_n = \binom{n+d-1}{n} \int_{\mathcal{U}_d} dU \, (U|\varphi\rangle\langle\varphi|U^\dagger)^{\otimes n}$$

is equal to the projector onto the symmetric subspace.

Proof: This follows from showing that

a) $\Pi_n \in \mathrm{End}(\mathrm{Sym}^n(\mathbb{C}^d))$

b) $\mathcal{U}_d$ acts irreducibly on $\mathrm{Sym}^n(\mathbb{C}^d)$ via $U \mapsto U^{\otimes n}$

c) Π_n is invariant under this action and Schur's lemma.

Details are left as an exercise. □

Recall the trace distance $D(\rho,\sigma) := \frac{1}{2}\|\rho-\sigma\|_1$

(with the trace norm $\|X\|_1 = \mathrm{tr}\sqrt{X^\dagger X}$),

and the fidelity $F(\rho,\sigma) = \|\sqrt{\rho}\sqrt{\sigma}\|_1$, and $D \leq \sqrt{1-F^2}$.

Prop (De Finetti theorem for pure symmetric states)

Let $\mathcal{H}_{A_i} \cong \mathbb{C}^d$ and $|\psi\rangle_{A_1 \ldots A_n} \in \mathrm{Sym}^n(\mathbb{C}^d)$.

Then for any $k < n$,

$$D\left(\psi_{A_1 \ldots A_k}, \int d\varphi\, p(\varphi)\, |\varphi\rangle\langle\varphi|^{\otimes k}\right) \leq \sqrt{\frac{dk}{n-k}}.$$

Here, $d\varphi$ is the measure on pure states induced by the Haar measure, i.e., $|\varphi\rangle = U|\varphi_0\rangle$ for some fixed $|\varphi_0\rangle$ and $d\varphi\, f(|\varphi\rangle) = dU\, f(U|\varphi\rangle)$.

$p(\varphi)$ is a probabality density that depends on $|\psi\rangle$.

<u>Proof:</u> The main idea is the following: For $m = n-k$ interpret $\Pi_m = \binom{m+d-1}{m} \int d\varphi\, |\varphi\rangle\langle\varphi|^{\otimes m}$ as a continuous POVM with operators $\binom{m+d-1}{m} |\varphi\rangle\langle\varphi|^{\otimes m}$ measuring the last m systems that we trace out. Getting a specific outcome $|\varphi\rangle \in \mathbb{C}^d$ should then imply that the first k systems are also in the state $|\varphi\rangle^{\otimes k}$ on average, due to the permutation invariance of $|\psi\rangle_{A_1 \ldots A_n}$.

Since $|\psi\rangle_{A_1\dots A_n}$ is symmetric under arbitrary perm.s, in particular $|\psi\rangle_{A_1\dots A_n} = (\mathbb{1}_k \otimes \Pi_m)|\psi\rangle_{A_1\dots A_n}$ (recall $n = k+m$). Hence,

$$\psi_{A_1\dots A_k} = \mathrm{tr}_{A_{k+1}\dots A_n} |\psi\rangle\langle\psi|_{A_1\dots A_n}$$

$$= \mathrm{tr}_{A_{k+1}\dots A_n}\left[(\mathbb{1}_k \otimes \Pi_m)|\psi\rangle\langle\psi|_{A_1\dots A_n}\right]$$

$$= \binom{m+d-1}{m} \int d\varphi \, (\mathbb{1}_k \otimes \langle\varphi|^{\otimes m})|\psi\rangle\langle\psi|(\mathbb{1}_k \otimes |\varphi\rangle^{\otimes m})$$

where we used the partial cyclicity property

$$\mathrm{tr}_2((\mathbb{1}\otimes X_2)Y_{12}) = \mathrm{tr}_2(Y_{12}(\mathbb{1}\otimes X_2))$$

in the last equality. We define

$$\sqrt{p(\varphi)}\,|e_\varphi\rangle = \binom{m+d-1}{m}^{1/2} (\mathbb{1}_k \otimes \langle\varphi|^{\otimes m})|\psi\rangle_{A_1\dots A_n} \in (\mathbb{C}^d)^{\otimes k}$$

where $p(\varphi) \geq 0$ ensures that $\langle e_\varphi|e_\varphi\rangle = 1$.

Note that $p(\varphi)$ is a probability density, i.e., $\int d\varphi \, p(\varphi) = 1$.

Hence, we have $\psi_{A_1\dots A_k} = \int d\varphi \, p(\varphi)\, |e_\varphi\rangle\langle e_\varphi|$.

To show: $\int d\varphi \, p(\varphi)\, |e_\varphi\rangle\langle e_\varphi| \approx \int d\varphi \, p(\varphi)\, |\varphi\rangle\langle\varphi|^{\otimes k}$.

First, we compute the average (squared) fidelity of $|e_\varphi\rangle$ and $|\varphi\rangle^{\otimes k}$:

$\propto (\mathbb{1}_k \otimes \langle\varphi|^{\otimes m})|\psi\rangle$

$$
\begin{aligned}
\int d\varphi\, p(\varphi)\, F(|e_\varphi\rangle, |\varphi\rangle^{\otimes k})^2
&= \int d\varphi\, p(\varphi)\, \langle e_\varphi | \varphi^{\otimes k} | e_\varphi\rangle \\
&= \binom{m+d-1}{m} \int d\varphi\, \langle \psi | \varphi^{\otimes k+m} | \psi\rangle \\
&= \binom{m+d-1}{m} \cdot \binom{n+d-1}{n}^{-1} \underbrace{\langle \psi | \Pi_{k+m} | \psi\rangle}_{=1} \\
&= \binom{m+d-1}{m} \cdot \binom{k+m+d-1}{k+m}^{-1} \\
&= \frac{(m+d-1)!}{m!\,\cancel{(d-1)}!} \; \frac{(k+m)!\,\cancel{(d-1)}!}{(k+m+d-1)!} \\
&= \frac{(m+d-1)\dots(m+1)}{(k+m+d-1)\dots(k+m+1)} \\
&\geq \left(\frac{m+1}{k+m+1}\right)^{d-1} = \left(1 - \frac{k}{k+m+1}\right)^{d-1} \\
&\geq 1 - \frac{k(d-1)}{k+m+1} \geq 1 - \frac{kd}{m}
\end{aligned}
$$

Now we can finish the proof. Recall the Fuchs-van-de-Graaf inequality $D(\rho,\sigma) \le \sqrt{1-F(\rho,\sigma)^2}$ (which is actually an equality for pure states, see Ex.).

$$D\left(\psi_{A_1\ldots A_n}, \int d\varphi \, p(\varphi)\, |\varphi\rangle\langle\varphi|^{\otimes n}\right)$$

$$= D\left(\int d\varphi\, p(\varphi)\, |e_\varphi\rangle\langle e_\varphi|, \int d\varphi\, p(\varphi)\, |\varphi\rangle\langle\varphi|^{\otimes n}\right)$$

convexity of norms

$$\le \int d\varphi\, p(\varphi)\, D\left(|e_\varphi\rangle\langle e_\varphi|, |\varphi\rangle\langle\varphi|^{\otimes n}\right)$$

$$\le \int d\varphi\, p(\varphi) \sqrt{1 - F(e_\varphi, \varphi^{\otimes n})^2}$$

Jensen's ineq.

$$\le \left(\int d\varphi\, p(\varphi)\left(1 - F(e_\varphi, \varphi^{\otimes n})^2\right)\right)^{1/2}$$

$$= \left(1 - \int d\varphi\, p(\varphi)\, F(e_\varphi, \varphi^{\otimes n})^2\right)^{1/2}$$

$$\le \sqrt{1 - \left(1 - \frac{nd}{m}\right)} = \sqrt{\frac{nd}{m}}. \qquad \square$$

§ 7.3 Extension to permutation-invariant mixed states

A state $\rho_{A_1\ldots A_n}$ is called permutation-invariant if

$$Q_\pi\, \rho_{A_1\ldots A_n}\, Q_\pi^\dagger = \rho_{A_1\ldots A_n} \quad \text{for all } \pi \in S_n.$$

$(Q_\pi \equiv \varphi(\pi))$

Goal: prove de Finetti theorem for permutation-invariant ρ.

Strategy: use our theorem from §7.2 for pure states!

Need to relate perm.-inv. states to pure states in $\mathrm{Sym}^n(\mathbb{C}^d)$

$\boxed{\text{Lem}}$ Let $\mathcal{H}_{A_i} = \mathbb{C}^d$ for $i = 1, \ldots, n$ and $\rho_{A_1 \ldots A_n}$ be permutation-invariant. Then $\rho_{A_1 \ldots A_n}$ has a purification

$$|\psi^\rho\rangle \in \mathrm{Sym}^n(\mathbb{C}^d \otimes \mathbb{C}^d).$$

<u>Proof:</u> Let $\rho_{A_1 \ldots A_n} = \sum_{\lambda \in \mathrm{spec}(\rho)} \lambda P_\lambda$ be a spectral decomposition, where $\mathrm{spec}(\rho)$ is the set of distinct eigenvalues of ρ with corresponding orthogonal projector P_λ onto the eigenspace $\mathcal{H}_\lambda$. Since $\rho = Q_\pi \rho Q_\pi^\dagger$ for all $\pi \in S_n$, we have for any $\lambda \in \mathrm{spec}(\rho)$ and $|\phi\rangle \in \mathcal{H}_\lambda$ that

$$\lambda |\phi\rangle = \rho |\phi\rangle = Q_\pi \rho Q_\pi^\dagger |\phi\rangle$$

and hence $Q_\pi^\dagger |\phi\rangle \in \mathcal{H}_\lambda$ for all $\pi \in S_n$, i.e., the eigenspaces are permutation-invariant too, and $P_\lambda Q_\pi = Q_\pi P_\lambda$ for all $\pi \in S_n$, $\lambda \in \mathrm{spec}(\rho)$. Define $M = \sum_{\lambda \in \mathrm{spec}(\rho)} \sqrt{\lambda} P_\lambda$, then clearly also $Q_\pi M = M Q_\pi$ for all $\pi \in S_n$.

Let now $|\phi\rangle_{A_1\ldots A_n R_1\ldots R_n} := \sum_{x=1}^{d^n} |x\rangle_{A^n} \otimes |x\rangle_{R^n}$, where $\{|x\rangle\}_{x=1}^{d^n}$ is a product basis for $(\mathbb{C}^d)^{\otimes n}$, and set

$$|\psi^\rho\rangle = (M \otimes \mathbb{1}_{R^n})|\phi\rangle.$$

Then: ·) $\mathrm{tr}_{R^n} \psi^\rho_{A^n R^n} = M(\underbrace{\mathrm{tr}_{R^n} \phi_{A^n R^n}}_{= \mathbb{1}_{A^n}})M^\dagger$

$$= MM^\dagger = \sum_{\lambda,\lambda'} \sqrt{\lambda\lambda'}\, \underbrace{P_\lambda P_{\lambda'}}_{= \delta_{\lambda\lambda'} P_\lambda} = \rho_{A^n}$$

·) Note: If S_n acts on both $A_1\ldots A_n$ and $R_1\ldots R_n$ via Q_π, then S_n acts on $A_1\ldots A_n R_1\ldots R_n \cong A_1 R_1 \ldots A_n R_n$ via $Q_\pi \otimes Q_\pi$.

$$\Rightarrow (Q_\pi \otimes Q_\pi)|\psi^\rho\rangle = (Q_\pi \otimes Q_\pi)(M \otimes \mathbb{1})|\phi\rangle$$

$$= (Q_\pi M \otimes \mathbb{1})(\mathbb{1} \otimes Q_\pi)|\phi\rangle$$

transpose trick $\rightarrow$ $= (Q_\pi M Q_\pi^T \otimes \mathbb{1})|\phi\rangle$

$[M, Q_\pi] = 0$ $\rightarrow$ $= (M Q_\pi Q_\pi^T \otimes \mathbb{1})|\phi\rangle$

$Q_\pi^T = Q_\pi^\dagger$ $\rightarrow$ $= (M \otimes \mathbb{1})|\phi\rangle$

$$= |\psi^\rho\rangle \quad \text{for all } \pi \in S_n$$

$$\Rightarrow |\psi^\rho\rangle \in \mathrm{Sym}^n(\mathbb{C}^d \otimes \mathbb{C}^d). \quad \square$$

Prop Let $\mathcal{H}_{A_i} = \mathbb{C}^d$ for $i=1,\dots,n$ and $\rho_{A_1\dots A_n}$ be a permutation-invariant state. Then for any $k<n$,

$$D\left(\rho_{A_1\dots A_k}, \int d\mu(\sigma)\, \sigma^{\otimes k}\right) \leq \sqrt{\frac{d^2 k}{n-k}},$$

where $d\mu(\sigma)$ is a measure on the space of mixed states on $\mathbb{C}^d$ that depends on ρ.

<u>Proof:</u> Let $|\psi^\rho\rangle_{A^n R^n} \in \mathrm{Sym}^n(\mathbb{C}^d \otimes \mathbb{C}^d)$ be a symmetric purification of ρ. Applying the sym. subspace de Finetti thm shows that

$$D\left(\psi^\rho_{A_1R_1\dots A_kR_k}, \int d\varphi\, p(\varphi)\, |\varphi\rangle\langle\varphi|_{AR}^{\otimes k}\right) \leq \sqrt{\frac{d^2 k}{n-k}}$$

for a suitable prob. density $p(\varphi)$. The claim now follows from the monotonicity of $D(.,.)$ under partial trace:

$$D\left(\rho_{A_1\dots A_k}, \int d\varphi\, p(\varphi)\, \mathrm{tr}_R\, \varphi_{AR}^{\otimes k}\right)$$

$$\leq D\left(\psi^\rho_{A_1R_1\dots A_kR_k}, \int d\varphi\, p(\varphi)\, |\varphi\rangle\langle\varphi|_{AR}^{\otimes k}\right)$$

$$\leq \sqrt{\frac{d^2 k}{n-k}}. \qquad \square$$